Isocurvature constraints and Gravitational Ward Identity

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THE UNIVERSITY of WISCONSIN MADISON

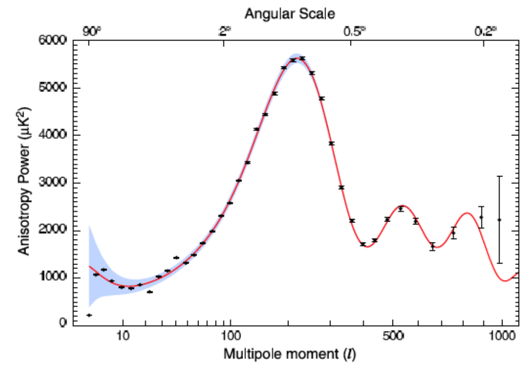
Outline

- Motivation
  - Non-Gaussianities
  - CDM Isocurvature models: Axion and WIMPzilla
- Isocurvature constraints
  - Relationship with NG
- "Gravitational Ward Identity"
- Application
  - Correlation between curvature and isocurvature

Primordial density perturbation

Evolution of Perturbations

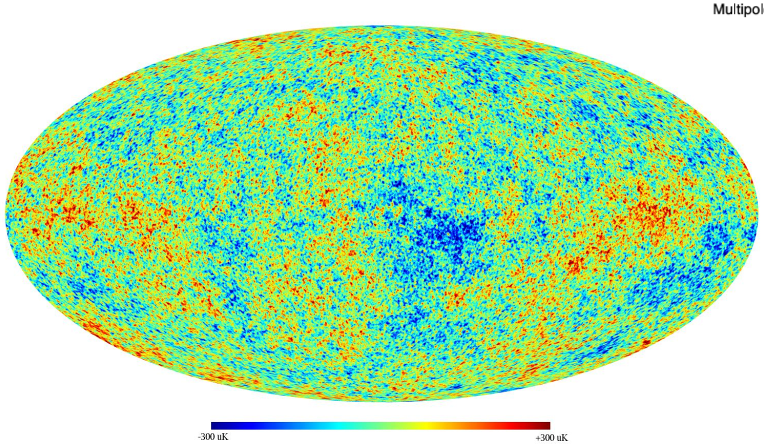
Hydrodynamics Equation
(Einstein + Boltzmann eqns)



Consistent with

the standard single field inflation model prediction:

- Scale Invariant
- Gaussian
- Adiabatic



Non-Gaussianities

- PLANCK can detect large non-Gaussianities
  - If detected, it will rule out single-field inflationary models. Maldacena(02), Seery et al., (05)

$$f_{NL} \sim \frac{\left\langle \frac{\Delta T}{T} \frac{\Delta T}{T} \frac{\Delta T}{T} \right\rangle}{\left\langle \frac{\Delta T}{T} \frac{\Delta T}{T} \right\rangle^2} \sim O(\epsilon)$$

for single-field inf. models

  - It can be a probe of new physics

- A possible way to generate large "local" non-Gaussianities
  - Additional degree of freedom during inflation

Additional Scalar field σ

Density Perturbation $$\frac{\delta\rho_\sigma}{\rho_\sigma} \sim \cancel{\frac{2\delta\sigma}{\sigma}} + \frac{\delta\sigma^2}{\sigma^2} + \cdots$$

Non-Gaussian terms

The linear term vanishes if background $\bar{\sigma} = 0$
the density perturbation becomes non-Gaussian.

$$\frac{\delta\rho_\sigma}{\rho_\sigma}$$

Unstable σ
Decays into radiation

$$\zeta \sim r\frac{\delta\rho_\sigma}{\rho_\sigma}$$

Curvaton models
Lyth&Wands,Moroi&Takahashi (01)

Stable σ
Remains as CDM

$$S \sim \frac{\delta\rho_\sigma}{\rho_\sigma}$$

CDM Isocurvature
Linde&Mukhanov (97)

CDM Isocurvature Models

Axion: To solve the strong CP problem Peccei,Quinn (77)

PQ sym. undergoes SSB before inflation and is never restored.

- $f_a > H_{inf}/2\pi$

Axions are produced by "vacuum misalignment" mechanism

- $\delta a = f_a \delta\theta = H_{inf}/2\pi$

Coherent oscillation after QCD phase transition behaves like CDM .

Super Heavy Fields: To avoid thermalization Chung,Kolb&Riotto(98)

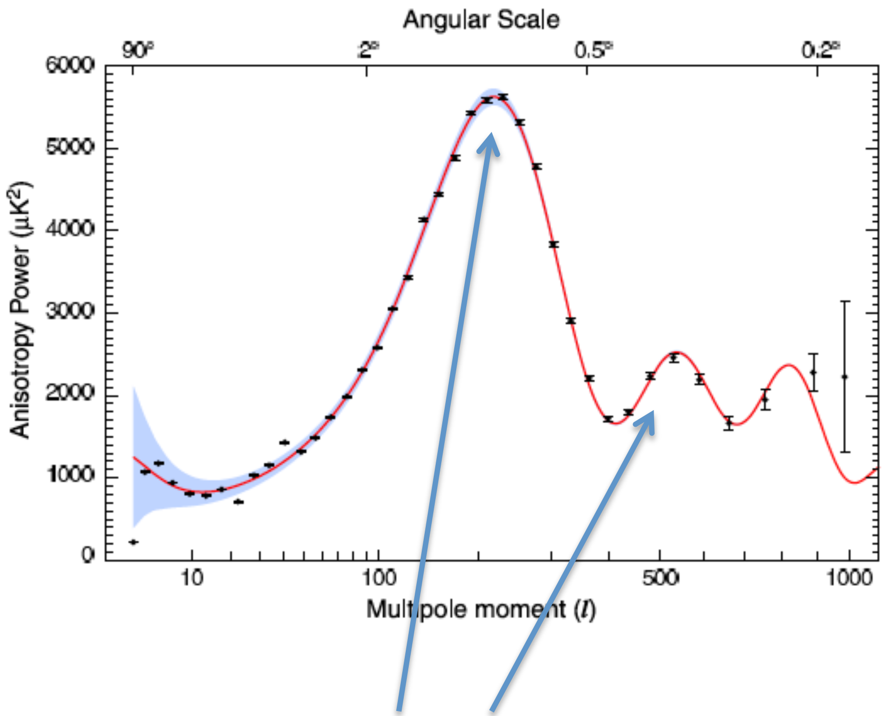
Super heavy stable particles are non-thermal DM called "WIMPzillas"

- $\langle\sigma v\rangle \leq 1 / m_X^2$
- For large m_X,

  $n\langle\sigma v\rangle \ll H$.

Many candidates motivated by SUSY and String theory exists: Q-Balls, D-Matter, Crypton, light KK particles …

CDM Isocurvature constraints



Consistent with adiabatic init. condition,

But admixture with isocurvature
is not ruled out!

Adiabaticity Parameters

$$\alpha = \frac{\langle SS \rangle}{\langle \zeta\zeta \rangle + \langle SS \rangle}$$

$$\beta = \frac{\langle \zeta S \rangle}{\sqrt{\langle \zeta\zeta \rangle \langle SS \rangle}}$$

Observational Bounds

Komatsu(10), Sollom(09)

($n_s \approx 1$)

$$\alpha_0 \lesssim 0.07 \quad \text{for } \beta = 0$$

$$\alpha_{-1} \lesssim 0.004 \quad \text{for } \beta = -1$$

Significantly different!

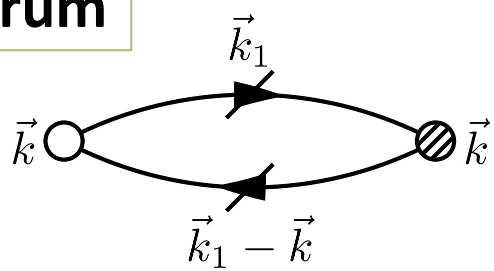
Quadratic type

$$S = \omega_\sigma \frac{\delta\sigma^2}{\sigma^2} \rightarrow \hat{S} = \omega_\sigma \frac{\hat{\sigma}^2 - \langle\hat{\sigma}^2\rangle}{\langle\hat{\sigma}^2\rangle} \quad \text{where} \quad \omega_\sigma = \frac{\rho_\sigma}{\rho_{CDM}}$$

In the long-wavelength limit

$$P_\sigma(k) = \frac{\langle\hat{\sigma}_{\vec{k}}\hat{\sigma}_{-\vec{k}}\rangle}{\langle\hat{\sigma}^2\rangle} \approx \frac{A}{k^3}\left[\frac{k}{k_0}\right]^n \qquad \text{where} \qquad n = 3 - 2\sqrt{\frac{9}{4} - \frac{m_\sigma^2}{H^2}}$$

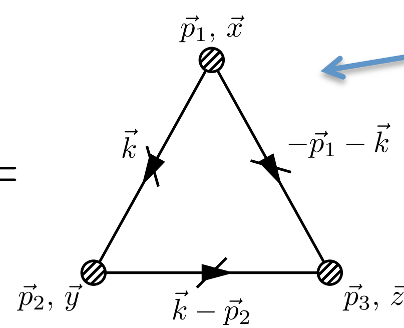
Power spectrum

$\Delta_S^2(k) =$ 

$\approx \frac{k^3}{2\pi^2} 2\omega_\sigma^2 \int_{\Lambda_{IR}}^{a_e H} [d^3k_1] P_\sigma(k) P_\sigma(|\vec{k} - \vec{k}_1|)$

Bi-spectrum

Non-Gaussianity

$B_S(\vec{p}_1, \vec{p}_2, \vec{p}_3) =$ 

$\approx 8\omega_\sigma^3 \int_{\Lambda_{IR}}^{a_e H} [d^3k] P_\sigma(k) P_\sigma(|\vec{p}_1 + \vec{k}|) P_\sigma(|\vec{p}_2 - \vec{k}|)$

$$\Rightarrow B_S \propto \Delta_S^{3/2}$$

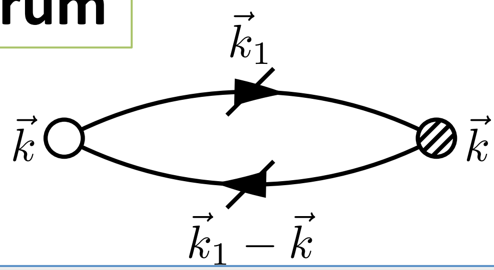
The squeezed conf.

when $p_1,\ p_2 \gg p_3$

Properties of Δ_S and B_S

Power spectrum

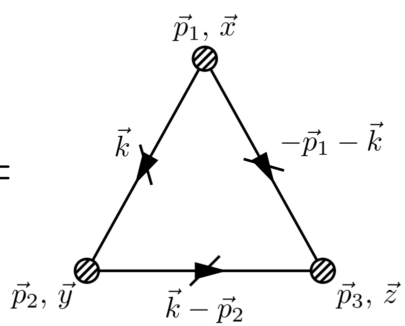
$$\Delta_S^2(k) =$$



Almost arbitrarily tunable using parameters: $\omega_\sigma(H, T_{RH}, f_a, \ldots)$, $m^2/H^2, \ldots$

Bi-spectrum

$$B_S(\vec{p}_1, \vec{p}_2, \vec{p}_3) =$$



$$\propto \Delta_S^{3/2}$$

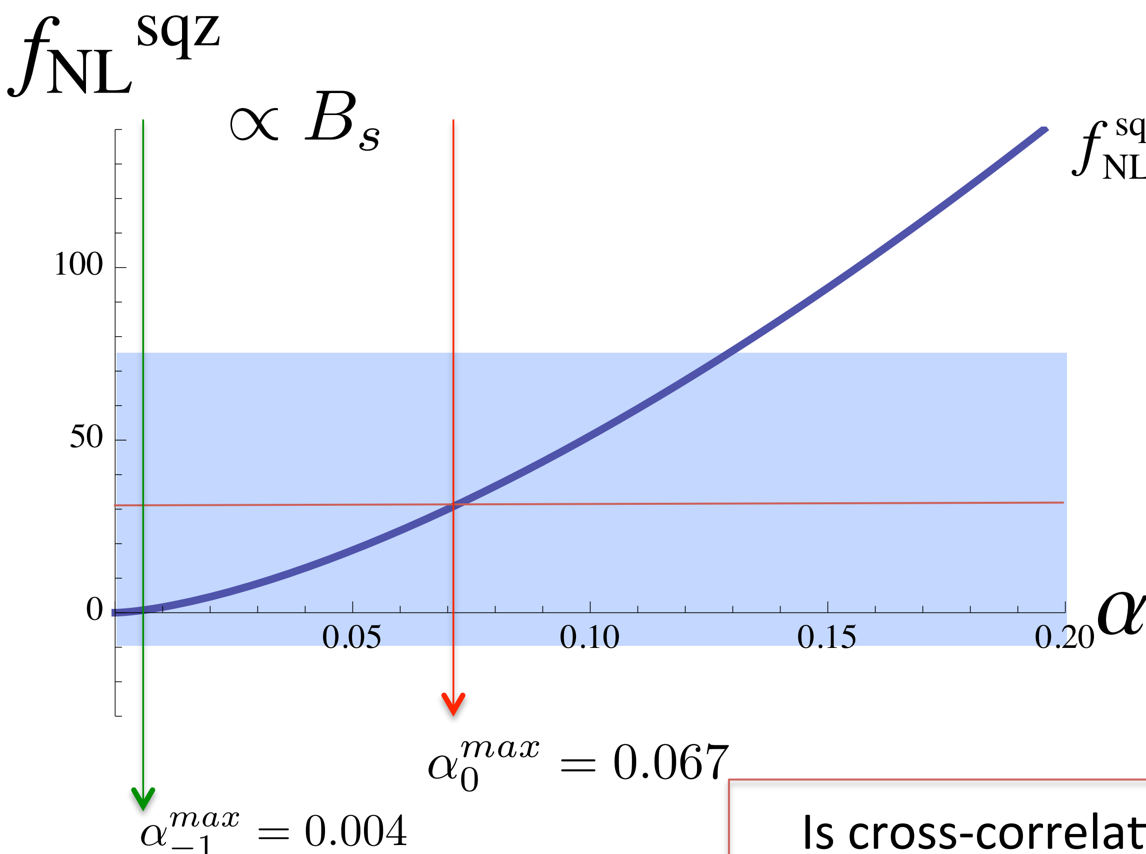
Completely fixed by the power spectrum

- For maximum NG, the isocurvature amplitude should be saturated to its observational upper bound.

Chung&Yoo(11)

NG by CDM Isocurvature

Hikage et al. (09)
Chung&Yoo (11)



$$f_{NL}^{sqz} \approx 30\left(\frac{\alpha}{0.07}\right)^{3/2}$$

WMAP Hint

$-10 < f_{NL}^{sqz} < 75$
(95% C.L.)

Is cross-correlation β large or small?

$$\beta = \frac{\langle \zeta S \rangle}{\sqrt{\langle \zeta\zeta \rangle \langle SS \rangle}} =?$$

Naïve Intuition for small cross-correlation β << 1

$$S \to \zeta$$

- Scalar field σ is an energetically subdominant component during inflation. σ cannot generate large curvature perturbation ζ.

$$\frac{\rho_\sigma}{\rho_{total}} \lesssim \frac{H_{inf}^2}{M_p^2} \sim 10^{-10}$$

$$\zeta \to S$$

- Curvature perturbation ζ also cannot affect the gravitational particle production process, because of large separation of particle production and CMB scales.

$$k_{CMB}/k_{\mathrm{p.p.}} \sim e^{50}$$

Naïve Intuition
for small cross-correlation β << 1

$$S \to \zeta$$

- Scalar field σ is an energetically subdominant component during inflation. σ cannot generate large curvature perturbation ζ. $\frac{\rho_\sigma}{\ } \lesssim \frac{H_{inf}^2}{M^2} \sim 10^{-10}$

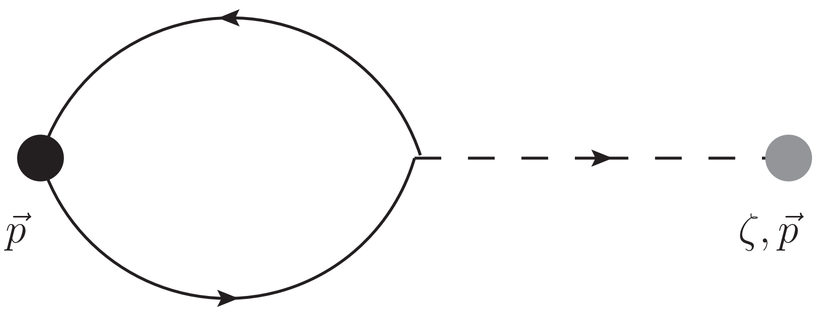
This has NOT been proven!

$$\zeta \to S$$

- Curvature perturbation ζ also cannot affect the gravitational particle production process, because of large separation of particle production and CMB scales.

$$k_{CMB}/k_{\text{p.p.}} \sim e^{50}$$

Explicit computation $\left\langle \hat{S}\hat{\zeta} \right\rangle$



$$S_{int}^{(C)} = \int d^4x\, a^3 \left[T^{ij} a^2 \delta_{ij} \zeta + T^{0i} \partial_i \left(-\frac{\zeta}{H} + \epsilon \frac{a^2}{\nabla^2} \dot{\zeta} \right) - T^{00} \frac{\dot{\zeta}}{H} \right]$$

$$\ni -\int d^4x\, a^3 \frac{3}{2} m_\sigma^2 \sigma^2 \zeta$$

Dangerous Interaction terms yield strong cross-correlation

$$\left\langle \hat{S}\hat{\zeta} \right\rangle \sim O(1) \left\langle \hat{\zeta}\hat{\zeta} \right\rangle$$

$$\rightarrow \beta \sim O(1) \text{ for massive } \sigma$$

What is wrong?

Further Questions

- Gauge Invariance of the correlators

$$\langle \zeta\zeta \rangle\,,\ \langle SS \rangle\,,\ \langle \zeta S \rangle \cdots$$

  – ζ, S are gauge invariant variables.
  – Difficulties:
    - σ has no classical background.
    - S is a composite operator.

These questions are answered by
"Gravitational (Diffeomorphism) Ward Identity"

“Gravitational” Ward Identity

Under $x^\mu \to \tilde{x}^\mu = x^\mu - \epsilon^\mu$,the metric transforms as

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} + \mathcal{L}_\epsilon g_{\mu\nu} = g_{\mu\nu} + 2\nabla_{(\mu}\epsilon_{\nu)}.$$

Whereas the two-point function should remain invariant

$$\begin{aligned} \langle\phi(x)\phi(y)\rangle_g &= \langle\phi(\tilde{x})\phi(\tilde{y})\rangle_{\tilde{g}} = \int \mathcal{D}\phi e^{iS[\phi;\tilde{g}]}\phi(\tilde{x})\phi(\tilde{y}) \\ &= \int \mathcal{D}\phi e^{iS[\phi;g+\mathcal{L}_\epsilon g]}\phi(x-\epsilon)\phi(y-\epsilon). \end{aligned}$$

Expand w.r.t ϵ.

$$\frac{i}{2}\int d^4z\sqrt{-g_z}\left(\nabla_\mu\epsilon_\nu\right)_z \langle T_z^{\mu\nu}\phi_x\phi_y\rangle_g = \epsilon^\mu \left\langle \frac{\partial}{\partial x^\mu}\phi_x\phi_y \right\rangle_g + \epsilon^\mu \left\langle \phi_x\frac{\partial}{\partial y^\mu}\phi_y \right\rangle_g .$$

Equivalently,

$$i\sqrt{-g_z}\nabla_{\mu,z}\langle T_z^{\mu\nu}\phi_x\phi_y\rangle_g = \delta^4(x-z)\left\langle \frac{\partial}{\partial x^\mu}\phi_x\phi_y \right\rangle_g + \delta^4(x-z)\left\langle \phi_x\frac{\partial}{\partial y^\mu}\phi_y \right\rangle_g .$$

In-In formalism

Weinberg(05)

$$\left\langle in \left| \hat{\mathcal{O}}(t) \right| in \right\rangle = \left\langle \left(T e^{-i\int_{-\infty}^{t} \hat{H}^I{}_{int}(t')dt'} \right)^{\dagger} \hat{\mathcal{O}}^I(t) \left(T e^{-i\int_{-\infty}^{t} \hat{H}^I{}_{int}(t'')dt''} \right) \right\rangle$$

$$= \left\langle \hat{\mathcal{O}}^I(t) \right\rangle + (-i)\int_{-\infty}^{t} dt' \left\langle \left[\hat{\mathcal{O}}^I(t), \hat{H}^I{}_{int}(t') \right] \right\rangle + \cdots$$

Interaction vertices come with commutators

Apply it to cross-correlator in the comoving gauge

$$\left\langle in \left| \hat{S}(t,\vec{x})\hat{\zeta}(t,\vec{y}) \right| in \right\rangle = \frac{1}{\langle \hat{\sigma}^2 \rangle} i \int^{t} dt_z d^3 z\, a_z^3 \left\langle \left[\hat{\sigma}_x^2 \hat{\zeta}_y, \frac{1}{2} \left(\hat{T}_{\sigma}^{\mu\nu} \delta\hat{g}_{\mu\nu} \right)_z \right] \right\rangle$$

where $$\delta g_{\mu\nu} = \begin{pmatrix} -2\frac{\dot{\zeta}}{H} & (-\frac{\zeta}{H} + \epsilon\frac{a^2}{\nabla^2}\dot{\zeta})_{,i} \\ (-\frac{\zeta}{H} + \epsilon\frac{a^2}{\nabla^2}\dot{\zeta})_{,i} & a^2\delta_{ij}2\zeta \end{pmatrix}$$

The metric perturbation is obtained by ADM formalism
With gauge fixed.

Feynman Diagram for In-In formalism

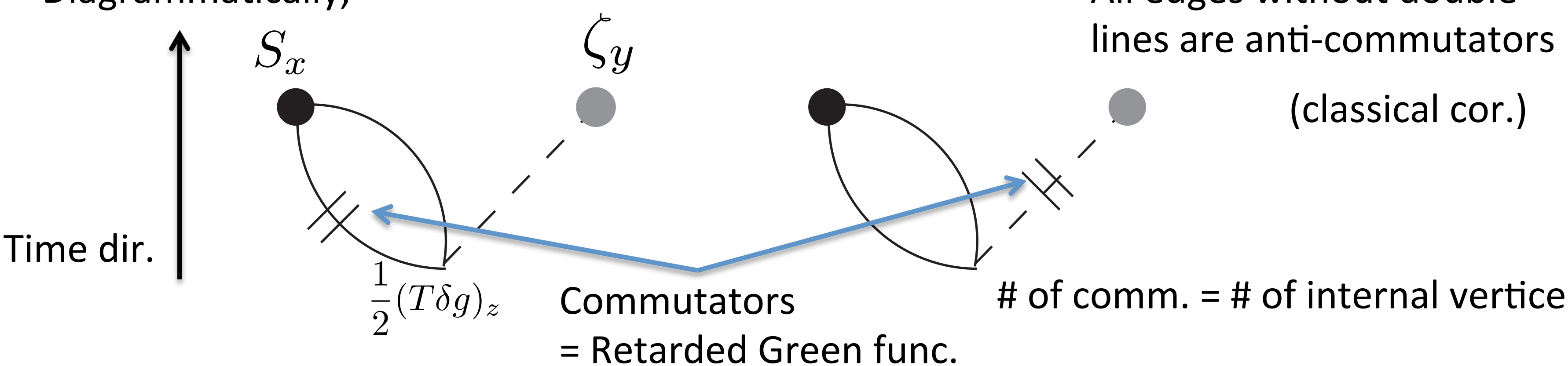
Commutator and anti-commutator decomposition:

- Easy to give physical interpretation
- Convenient to count the power of scale factor "a"

For example,

$$\left\langle \hat{S}_x \hat{\zeta}_y \right\rangle = \frac{1}{\langle \hat{\sigma}^2 \rangle} i \int d^4 z \, a_z^3 \left\langle \left[\hat{\sigma}_x^2, \frac{1}{2} \hat{T}^{\mu\nu}_{\sigma,z} \right] \right\rangle \left\langle \left\{ \hat{\zeta}_y, \delta \hat{g}_{\mu\nu,z} \right\} \right\rangle + \frac{1}{\langle \hat{\sigma}^2 \rangle} i \int d^4 z \, a_z^3 \left\langle \left\{ \hat{\sigma}_x^2, \frac{1}{2} \hat{T}^{\mu\nu}_{\sigma,z} \right\} \right\rangle \left\langle \left[\hat{\zeta}_y, \delta \hat{g}_{\mu\nu,z} \right] \right\rangle$$

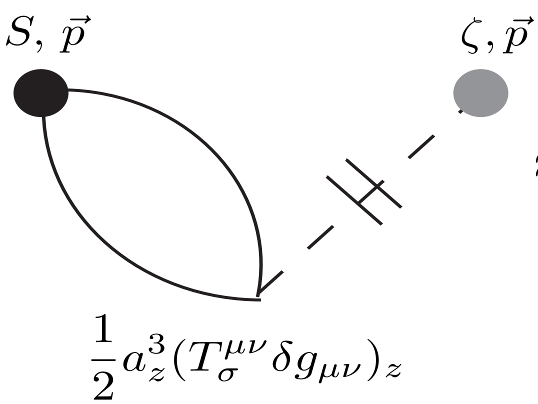
Diagrammatically,



Curvature induces isocurvature

Isocurvature induces curvature

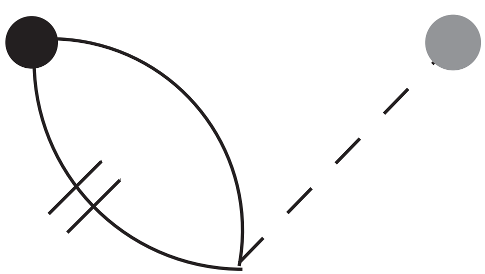
Compute Cross-correlation



$$\approx -\frac{1}{12\pi^2}\frac{\left\langle \hat{\zeta}_{\vec{p}}\hat{\zeta}_{-\vec{p}}\right\rangle}{\langle \hat{\sigma}^2\rangle}H^2\left(\frac{p}{aH}\right)^{3-2\nu}$$

where $\quad 3-2\nu \approx \frac{2}{3}\frac{m_\sigma^2}{H^2}+O(\frac{m_\sigma^4}{H^4})$

With the long wavelength approximation
(Only super horizon modes are taken in the loop)

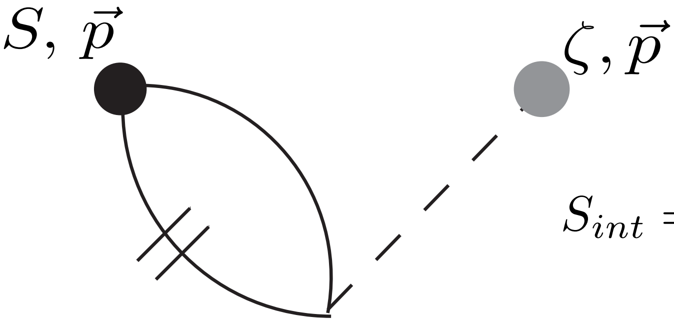


This diagram generates strong cross-correlation.

- It suffers from UV divergence in the loop, which signals the long wavelength approximation is not reliable.

- The UV cut-off regulator with the long wavelength approximation obscures possible cancellation due to symmetry.

Dangerous Interaction term



$$S_{int} = \int d^4x a_x^3 [T^{ij} a^2 \delta_{ij} \zeta + T^{0i}(-\frac{\zeta}{H} + \epsilon \frac{a^2}{\nabla^2}\dot{\zeta})_{,i} - T^{00}\frac{\dot{\zeta}}{H}]$$

This is the dangerous interaction term.

Non-derivative coupling

Explicitly,

$$I^{ij}_{\zeta \to S}(p) \approx \frac{\langle \hat{\zeta}_{\vec{p}} \hat{\zeta}_{-\vec{p}} \rangle}{\langle \hat{\sigma}^2 \rangle}\, i \int_{t_p}^{t} dt_z d^3 z a_z^3 e^{-i\vec{p}\cdot\vec{z}} \langle [\hat{\sigma}^2(\vec{0}, t), \hat{T}^{ij}(z)] \rangle \, (a^2 \delta_{ij})$$

Hinterbichler,Hui&Khoury(12)

Note that $"\zeta = \mathrm{const}"$ is a pure gauge. i.e. rescaling scale factor $a \to a e^{\zeta}$

Thus, we expect $\lim_{p \to 0} I^{ij}_{\zeta \to S}(p) = 0 \ \Rightarrow I^{ij}_{\zeta \to S}(p) \propto O(p^2/a^2)$

Proof by Ward Identity : $\lim_{p\to 0} I^{ij}_{\zeta\to S}(p) = 0$.

Chung,Yoo&Zhou(in prep)

Ward Identity (in-in version) for σ^2

$$i\int^t dt_z d^3z\, a_z^3\, (\nabla_\mu \epsilon_\nu)_z \left\langle \left[T_z^{\mu\nu}, \sigma^2(x)\right]\right\rangle = \epsilon^\mu \left\langle \frac{\partial}{\partial x^\mu}\sigma^2(x)\right\rangle$$

(requires careful regulator choice.)

Choose $\epsilon^\mu = (0, \vec{x})$ spatial dilatation flow

(For a rigorous proof,
a window function may be needed.)

$$\nabla_i \epsilon_j = a^2 \delta_{ij} \qquad \nabla_\mu \epsilon_0 = \nabla_0 \epsilon_\nu = 0.$$

RHS vanishes because of translational symmetry of the FRW space-time.

$$\Rightarrow \int^t dt_z d^3z\, a_z^3 \left\langle \left[T_z^{ij}\delta_{ij} a_z^2, \sigma^2(x)\right]\right\rangle = 0$$

Summary of Computation

$$\langle \zeta\zeta \rangle = \frac{H^2}{4\epsilon M_p^2 p^3}$$

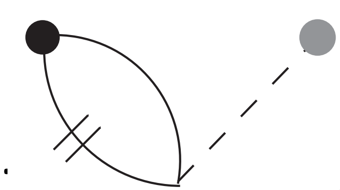
$$\langle SS \rangle \sim \frac{3p^{-3}}{2\pi^2}\frac{H^6}{m^2(\bar{\sigma}^2)^2}\left(\frac{p}{a_x H}\right)^{6-4\nu}\left(1-\left(\frac{\Lambda_{IR}}{p}\right)^{3-2\nu}\right)$$

$$\langle S\zeta \rangle \sim \frac{1}{12\pi^2}\frac{\langle \zeta\zeta \rangle}{\langle \sigma^2 \rangle}H^2\left(\frac{p}{aH}\right)^{3-2\nu}$$

$$\Rightarrow \beta = \frac{\langle S\zeta \rangle}{\sqrt{\langle \zeta\zeta \rangle \langle SS \rangle}} \sim \frac{1}{8}\sqrt{\frac{2}{3}}\frac{m_\sigma}{M_p\sqrt{\epsilon}}\left(1-\left(\frac{\Lambda_{IR}}{p}\right)^{3-2\nu}\right)^{-1/2} \ll 1$$

Caveat:

- IR mode contribution is neglected.
- Numerical O(1) factor cannot be estimated accurately.

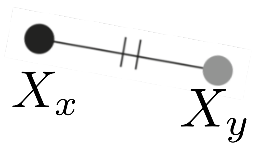


Conclusion

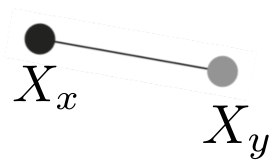
- Gravitational Ward Identity is very useful in perturbation theory.
  - provides a general proof of gauge invariance of correlation functions. $\langle\zeta\zeta\rangle,\ \langle SS\rangle,\ \langle S\zeta\rangle$
  - clarifies non-trivial cancellation due to diffeomorphism.
- Application: cross-correlation between ζ and S
  - We showed cross-correlation β is small for Axion/WIMPzilla isocurvature models. (The 1st proof)
    - These models can generate large non-Gaussianities.

Advantages of Decomposition

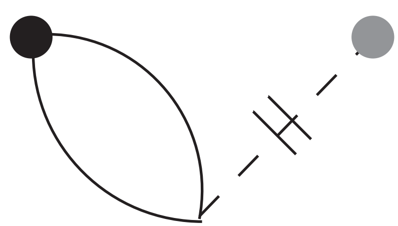
For super horizon mode



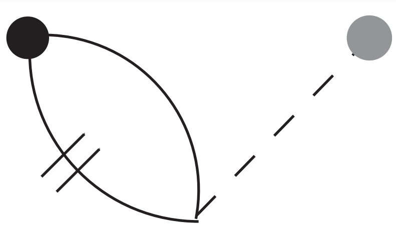
$$= G^X_{ret}(x,y) \propto \begin{cases} \frac{1}{a_y^3} & \text{for } \zeta \\ a_x^{-3/2+\nu} a_y^{-3/2-\nu} & \text{for } \sigma \end{cases}$$



$$\propto \begin{cases} a^0 \\ a_x^{-3/2+\nu} a_y^{-3/2+\nu} \end{cases}$$



$= I_{S\to\zeta}$ Isocurvature induces curvature



$= I_{\zeta\to S}$ Curvature induces isocurvature

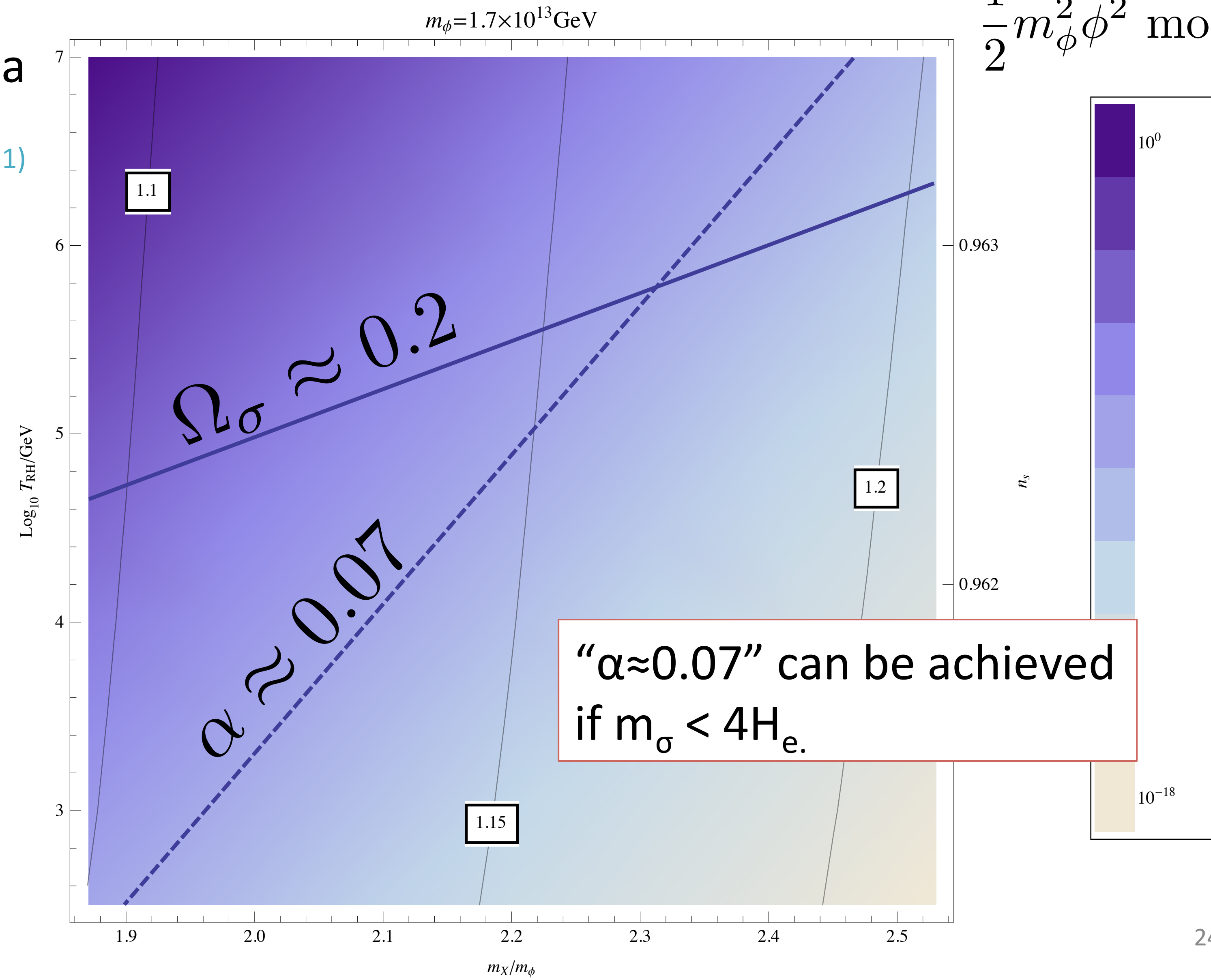
Large NG is achievable if a model can saturate the isocurvature power spectrum.

Conditions: $\alpha \sim 0.07, \quad \Omega_\sigma \lesssim 1$

WIMPZilla

Chung&Yoo(11)

$\frac{1}{2} m_\phi^2 \phi^2$ model



"α≈0.07" can be achieved if $m_\sigma < 4H_{e}$.

NG by CDM Isocurvature

Axion

- The average phase angle θ_i is negligible.
- During inflation, axion is effectively massless

$$\Rightarrow \Delta_S(t_e) \sim 1$$

At the end of inflation t_e

$$\Rightarrow \Delta_S(t_r) \sim \omega_a^2 \Delta_S(t_e) \sim \omega_a^2$$

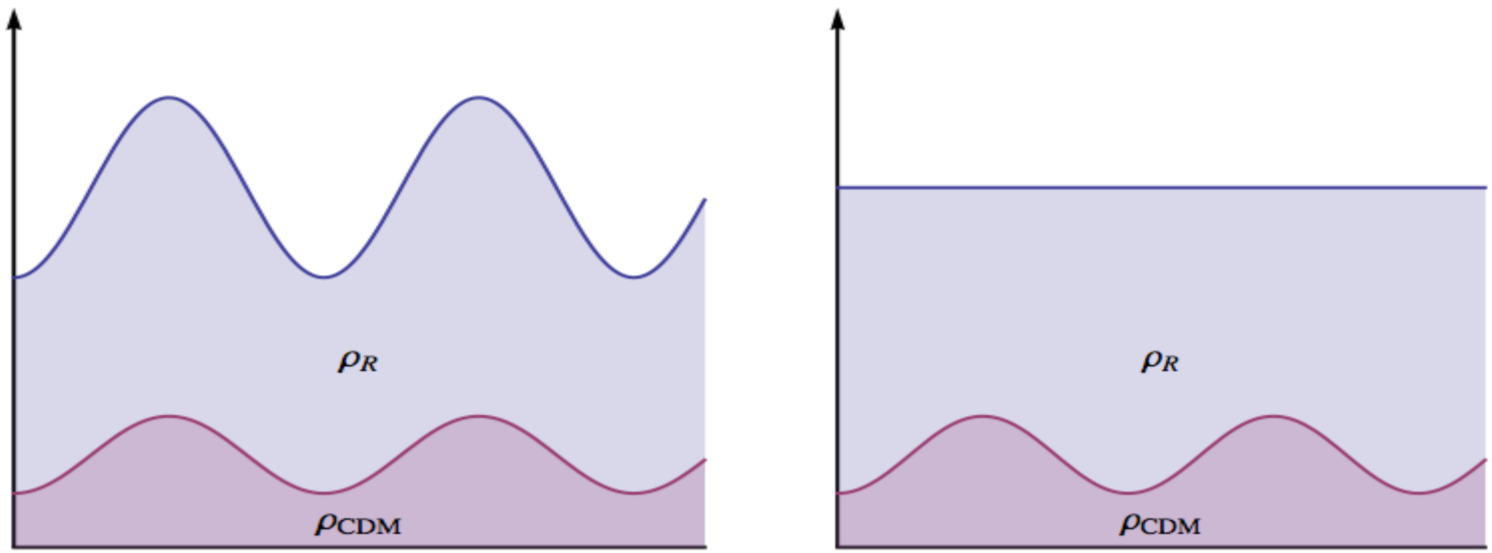
After QCD phase transition t_r

$$\Rightarrow \alpha = \frac{\omega_a^2}{\Delta_\zeta} \sim 10^{-9} \frac{\Omega_a^2}{\Omega_{CDM}^2} < 0.07 \quad \boxed{\Rightarrow H_{inf} \lesssim 2 \times 10^{10} \left(\frac{f_a}{10^{12}\,GeV} \right)^{0.41} GeV}$$

$$f_a \theta_i < \frac{H_{inf}}{2\pi} \Rightarrow \boxed{\theta_i < 3 \times 10^{-3} \left(\frac{f_a}{10^{12}\,GeV} \right)^{-0.59}}$$

Large NG is achievable for both CDM isocurvature models with underlying assumption: small cross-correlation $\beta << 1$

Adiabatic vs. Isocurvature



- Adiabatic

$$\frac{\delta\rho}{\rho+P}=\frac{\delta\rho_\gamma}{\rho_\gamma+P_\gamma}=\frac{\delta\rho_b}{\rho_b+P_b}=\frac{\delta\rho_{\mathrm{CDM}}}{\rho_{\mathrm{CDM}}+P_{\mathrm{CDM}}}=\cdots\neq 0$$

$$\zeta=-\Phi+\frac{\delta\rho}{3(\rho+P)}$$

- Isocurvature

$$\frac{\delta\rho}{\rho+P}=0,\quad \frac{\delta\rho_\gamma}{\rho_\gamma+P_\gamma}\neq\frac{\delta\rho_{\mathrm{CDM}}}{\rho_{\mathrm{CDM}}+P_{\mathrm{CDM}}}$$

$$S=\frac{\delta\rho_{\mathrm{CDM}}}{\rho_{\mathrm{CDM}}+P_{\mathrm{CDM}}}-\frac{\delta\rho_\gamma}{\rho_\gamma+P_\gamma}=\frac{\delta\rho_{\mathrm{CDM}}}{\rho_{\mathrm{CDM}}}-\frac{3\delta\rho_\gamma}{4\rho_\gamma}$$

For super horizon modes, $$\frac{\Delta T}{T}=-\frac{1}{5}\zeta-\frac{2}{5}S$$

Gauge Invariance

Under the coordinate change $x^\mu \to x^\mu - \epsilon^\nu$

Perturbations transform as

$$\delta Q \to \delta Q + \mathcal{L}_\epsilon \bar{Q}$$

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \mathcal{L}_\epsilon \bar{g}_{\mu\nu}$$

In general, perturbation is not invariant under the coordinate change.
However, some combinations of them do not depend on the coordinate.

For example,

$$\zeta = -\Psi + \frac{\delta\rho}{3(\rho + P)}, \qquad \zeta_X = -\Psi + \frac{\delta\rho_X}{3(\rho_X + P_X)}, \qquad S_X = 3(\zeta_X - \zeta),$$

where $g_{ij} = a^2(1 - 2\Psi)\delta_i j$

How can it apply to a quantum operator without VEV?

$$\hat{S}^{(C)} = \frac{\hat{\sigma}^2}{\langle\sigma^2\rangle}$$